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Weakly singular initial values for the Stokes equation on a half space

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Abstract

We study the extension of Ukai's formula to the case of singular initial values for the Stokes problem on the half space.

Keywords: Stokes equation; Besov spaces; Littlewood–Paley decomposition

0. Introduction

Following the book of Cannone [1], there has been a big amount of work on the existence of mild solutions to the Navier–Stokes problem on the space (recent results are reviewed in [4]). Those works culminated in the theorem of Koch and Tataru [3] dealing with initial values in BMO^{-1}.

If we want to adapt those results to the setting of a half space, the first step is to solve the Stokes equations associated to a singular initial value. We shall focus on Ukai's formula [5], which gives a solution associated to initial values in a Lebesgue space L^p. Cannone, Planchon and Schonbek [2] showed that the formula could be extended to initial values in

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Besov spaces $B_p^{s,\infty}$, $-1+1/p<s<0$. Ukai's analysis is based on the use of tangential Riesz transforms, which operate boundedly on Lebesgue spaces L^p and Besov spaces $B_p^{s,\infty}$ $(1<p<\infty)$. We shall slightly ameliorate those results by giving an alternative description of Ukai's formula which does not rely on the use of Riesz transforms. We shall mainly use the Littlewood–Paley decomposition, in the spirit of [1], in order to highlight the role of Besov spaces in estimating the size of the solutions.

Notations. Throughout the paper, n is the dimension of the space R^n and is assumed to satisfy $n\geqslant 3$.

For $x=(x',x_n)\in\mathbb{R}^n$, $x'=(x_1,\dots,x_{n-1})$ will be called the tangential part of x and x_n its normal coordinate.

We shall consider the following differential operators:

- the partial derivative operators $\partial_j=\frac{\partial}{\partial x_j}$ $(1\leqslant j\leqslant n)$, among which we have the tangential derivative operators $\partial_j'=\frac{\partial}{\partial x_j}$ $(1\leqslant j\leqslant n-1)$ and the normal derivative operator $\partial_n=\frac{\partial}{\partial x_n}$;
- the gradient operator $\vec{\nabla}=\begin{pmatrix}\partial_1\\ \vdots\\ \partial_n\end{pmatrix}$;
- the tangential gradient operator $\vec{\nabla}'=\begin{pmatrix}\partial_1\\ \vdots\\ \partial_{n-1}\end{pmatrix}$;
- the divergence operator $\nabla.\begin{pmatrix}f_1\\ \vdots\\ f_n\end{pmatrix}=\sum_{j=1}^n\partial_j f_j$;
- the tangential divergence operator $\nabla'.\begin{pmatrix}f_1\\ \vdots\\ f_{n-1}\end{pmatrix}=\sum_{j=1}^{n-1}\partial_j' f_j$;
- the Laplacian operator $\Delta=\sum_{j=1}^n\partial_j^2$;
- the tangential Laplacian operator $\Delta'=\sum_{j=1}^{n-1}\partial_j'^2$;
- the Fourier transform of a function $f(x)$ is the function $\mathcal{F}f=\hat{f}$ of the dual variable ξ defined by $\hat{f}(\xi)=\int_{\mathbb{R}^n}f(x)\,e^{-ix.\xi}\,dx$. We shall write $\xi=(\xi',\xi_n)$, where $\xi'\in\mathbb{R}^{n-1}$.

Moreover, throughout the paper, Ω will be the open half-space:

$$\Omega=\mathbb{R}_+^n=\left\{x=(x',x_n)\in\mathbb{R}^n\ \middle|\ x'\in\mathbb{R}^{n-1}\text{ and }x_n>0\right\}.$$

$\bar{\Omega}$ will be the closure of Ω and $\partial\Omega$ its border ($x\in\partial\Omega\Leftrightarrow x_n=0$).

1. Littlewood–Paley decomposition and Besov spaces

One of the main tools in this paper is the Littlewood–Paley decomposition of distributions into dyadic blocks of frequencies:

Definition 1 *(Dyadic blocks).* Let $\phi \in \mathcal{D}(\mathbb{R})$ be a non-negative function such that $|t| \leqslant \frac{1}{2} \Rightarrow \phi(t) = 1$ and $|t| \geqslant 1 \Rightarrow \phi(t) = 0$.

Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ be defined as $\varphi(\xi) = \phi(\xi_1) \dots \phi(\xi_n)$. We define the operators S_j and Δ_j as the Fourier multipliers $\mathcal{F}(S_j f) = \varphi(\xi/2^j)\mathcal{F}f$ and $\Delta_j = S_{j+1} - S_j$. The distribution $\Delta_j f$ is called the jth dyadic block of the Littlewood–Paley decomposition of f.

Similarly, let $\tilde{\varphi} \in \mathcal{D}(\mathbb{R}^{n-1})$ be defined as $\varphi(\xi') = \phi(\xi_1) \dots \phi(\xi_{n-1})$. We define the operators S'_j and Δ'_j as the Fourier multipliers on $\mathcal{S}'(\mathbb{R}^n)$ defined as $\mathcal{F}(S'_j f) = \varphi(\xi'/2^j)\mathcal{F}f$ and $\Delta'_j = S'_{j+1} - S'_j$. The distribution $\Delta'_j f$ is called the jth dyadic block of the tangential Littlewood–Paley decomposition of f.

We have the following classical result on the Littlewood–Paley decomposition of tempered distributions:

Lemma 1 *(Littlewood–Paley decomposition). For all $N \in \mathbb{Z}$ and all $f \in \mathcal{S}'(\mathbb{R}^n)$ we have*

$$f = S_N f + \sum_{j \geqslant N} \Delta_j f \quad \text{in } \mathcal{S}'\big(\mathbb{R}^n\big). \tag{1}$$

This equality is called the Littlewood–Paley decomposition of the distribution f.

Similarly, we have, for all $N \in \mathbb{Z}$ and all $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$f = S'_N f + \sum_{j \geqslant N} \Delta'_j f \quad \text{in } \mathcal{S}'\big(\mathbb{R}^n\big). \tag{2}$$

This equality will be called the tangential Littlewood–Paley decomposition of the distribution f.

Proof. Clearly, we have

$$\left\langle S_N f + \sum_{N \leqslant j < N+K} \Delta_j f \,\middle|\, g \right\rangle_{\mathcal{S}',\mathcal{S}} = \langle S_{N+K} f \mid g\rangle_{\mathcal{S}',\mathcal{S}} = \langle f \mid S_{N+K} g\rangle_{\mathcal{S}',\mathcal{S}}.$$

Thus, taking the Fourier transform $h = \hat{g}$ of g, it is enough to check that, for any $h \in \mathcal{S}(\mathbb{R}^d)$, we have $\lim_{N \to +\infty} \varphi(\xi/2^N)h(\xi) = h(\xi)$ strongly in $\mathcal{S}$. Similar proof holds for the tangential decomposition. □

Another important tool will be the use of Besov spaces $B_\infty^{\sigma,\infty}$:

Definition 2 *(Besov spaces).* For $\sigma \in \mathbb{R}$, the Besov space $B_\infty^{\sigma,\infty}(\mathbb{R}^n)$ is defined by

$$f \in B_\infty^{\sigma,\infty} \quad \Leftrightarrow \quad f \in \mathcal{S}'\big(\mathbb{R}^n\big), \quad S_0 f \in L^\infty \quad \text{and} \quad \sup_{j \geqslant 0} 2^{j\sigma} \|\Delta_j f\|_\infty < \infty$$

and is normed with

$$\|f\|_{B_\infty^{\sigma,\infty}} = \|S_0 f\|_\infty + \sup_{j \geqslant 0} 2^{j\sigma} \|\Delta_j f\|_\infty. \tag{3}$$

Equivalent norms will be useful for those Besov spaces. One checks easily the following classical result.

Lemma 2 *(Characterization of Besov spaces).*

(A) *For $\sigma < 0$, we have*

$$f \in B_\infty^{\sigma,\infty} \quad \Leftrightarrow \quad f \in \mathcal{S}'(\mathbb{R}^n) \quad \text{and} \quad \sup_{j \geqslant 0} 2^{j\sigma} \|S_j f\|_\infty < \infty.$$

(B) *For $0 < \sigma < 1$, f belongs to $B_\infty^{\sigma,\infty}$ if and only if $f \in \mathcal{C}^\sigma$, i.e., if and only if f is a continuous bounded function on $\mathbb{R}^n$ which is Hölderian with exponent σ:*

$$f \in B_\infty^{\sigma,\infty} \quad \Leftrightarrow \quad f \in L^\infty(\mathbb{R}^n) \quad \text{and} \quad \sup_{x\in\mathbb{R}^n,\, y\in\mathbb{R}^n,\, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\sigma} < \infty.$$

The role of the Besov spaces will be purely technical: we shall require some spatially uniform control on the distributions we shall deal with. More precisely, we have the following easy lemma.

Lemma 3. *Let τ_y be the translation operator $\tau_y f(x) = f(x-y)$. Then*

(A) *Let $\sigma \in \mathbb{R}$. Then we have for every $y \in \mathbb{R}^n$ and every $f \in B_\infty^{\sigma,\infty}$,*

$$\|\tau_y f\|_{B_\infty^{\sigma,\infty}} = \|f\|_{B_\infty^{\sigma,\infty}}.$$

(B) *Conversely, let E be a Banach space which is continuously included in $\mathcal{D}'(\mathbb{R}^n)$. If E is stable under the translations τ_y and if moreover, for some constant $A_0 \geqslant 1$, we have for every $y \in \mathbb{R}^n$ and every $f \in E$,*

$$A_0^{-1}\|f\|_E \leqslant \|\tau_y f\|_E \leqslant A_0\|f\|_E,$$

then E is embedded in $B_\infty^{\sigma,\infty}$ for some $\sigma \in \mathbb{R}$.

Proof. Point (A) is obvious. We prove point (B). Since E is embedded into $\mathcal{D}'(\mathbb{R}^n)$, there exists $N \in \mathbb{N}$ and $C_0 \geqslant 0$ such that for all $\gamma \in \mathcal{D}$ supported in the unit square $(-1,1)^n$ and all $f \in E$ we have

$$\big|\langle f \mid \gamma\rangle\big| \leqslant C_0\|f\|_E \sum_{|\alpha|\leqslant N} \big\|\partial^\alpha \gamma\big\|_\infty.$$

We easily check that for any $\gamma \in \mathcal{S}'(\mathbb{R}^n)$ we have

$$A(\gamma) = \sum_{k\in\mathbb{Z}^n} \sup_{y\in[-1,1]^n} \big|\gamma(y-k)\big| < \infty.$$

Now, we choose $\omega \in \mathcal{D}(\mathbb{R}^n)$ which is supported in $(-1,1)^n$ and satisfies $\sum_{k\in\mathbb{Z}^n} \omega(x-k) = 1$ and we write, for $\gamma \in \mathcal{D}(\mathbb{R}^n)$,

$$\langle f \mid \gamma\rangle = \sum_{k\in\mathbb{Z}^n} \big\langle f(y) \bigm| \omega(y-k)\gamma(y)\big\rangle = \sum_{k\in\mathbb{Z}^n} \big\langle f(y+k) \bigm| \omega(y)\gamma(y+k)\big\rangle,$$

hence

$$\big|\langle f \mid \gamma\rangle\big| \leqslant C_0 \sum_{k\in\mathbb{Z}^n} \big\|f(y+k)\big\|_E \sum_{|\alpha|\leqslant N} \sup_{y\in(-1,1)^n} \big|\partial_y^\alpha\big(\omega(y)\gamma(y+k)\big)\big|$$

and finally

$$\big|\langle f \mid \gamma\rangle\big| \leqslant C_1\|f\|_E \sum_{|\alpha|\leqslant N} A\big(\partial^\alpha\gamma\big).$$

Thus E is more precisely embedded into $\mathcal{S}'(\mathbb{R}^n)$.

Writing $f * \gamma(x) = \langle f(x+y) \mid \gamma(-y)\rangle$ and using the fact that $\|f(x+y)\|_E \leqslant A_0\|f(y)\|_E$, we find that, for $f \in E$ and $\gamma \in \mathcal{S}'$,

$$\|f*\gamma\|_\infty \leqslant A_0C_1\|f\|_E \sum_{|\alpha|\leqslant N} A\big(\partial^\alpha\gamma\big).$$

For $j \in \mathbb{N}$, we define γ as the inverse Fourier transform of φ and $\gamma_j(x) = 2^{jn}\gamma(2^jx)$, so that $S_jf = f * \gamma_j$. We find that

$$\begin{aligned}\|S_jf\|_\infty &\leqslant A_0C_1\|f\|_E \sum_{|\alpha|\leqslant N} A\big(\partial^\alpha\gamma_j\big)\\ &= A_0C_1\|f\|_E \sum_{|\alpha|\leqslant N} 2^{j(n+|\alpha|)} \sum_{k\in\mathbb{Z}^n} \sup_{y\in(-1,1)^n}\big|\partial^\alpha\gamma\big(2^j(y-k)\big)\big|.\end{aligned}$$

We write

$$\big|\partial^\alpha\gamma(x)\big| \leqslant C_\alpha\big(1+2|x|\big)^{-n-1}$$

hence we find that for $y \in (-1,1)^n$ and $j \in \mathbb{N}$ we have

$$\big|\partial^\alpha\gamma\big(2^j(y-k)\big)\big| \leqslant C_\alpha\begin{cases}1 & \text{if } |k|\leqslant 2,\\ k^{-n-1} & \text{if } |k|>2.\end{cases}$$

This gives that f belongs to $B_\infty^{-N-n,\infty}$. $\square$

Since we shall work on the half space, we shall require a different control on the normal x_n variable than for the tangential coordinates. More precisely, we introduce some modified Besov spaces:

Definition 3 *(Tangential Besov spaces).* For $\sigma \in \mathbb{R}$ and $k \in \mathbb{R}$, the tangential Besov space $B_\infty^{\sigma,k,\infty}$ is defined by

$$f \in B_\infty^{\sigma,k,\infty} \quad\Leftrightarrow\quad f \in \mathcal{S}'\big(\mathbb{R}^n\big) \quad\text{and}\quad \big(1+x_n^2\big)^{-k/2}f \in B_\infty^{\sigma,\infty}\big(\mathbb{R}^n\big)$$

and is normed with

$$\|f\|_{B_\infty^{\sigma,k,\infty}} = \big\|\big(1+x_n^2\big)^{-k/2}f\big\|_{B_\infty^{\sigma,\infty}}. \tag{4}$$

We shall deal with some useful equivalent norms for those tangential Besov spaces:

Lemma 4 *(Characterization of tangential Besov spaces).*

(A) *Let $\eta \in \mathcal{D}(\mathbb{R})$ be equal to* 1 *on* $[-1,1]$. *Then, for $\sigma \in \mathbb{R}$ and $k \geqslant 0$, we have*

$$f \in B_\infty^{\sigma,k,\infty} \quad\Leftrightarrow\quad \sup_{j\geqslant 0} 2^{-jk}\big\|\eta\big(x_n/2^j\big)f(x)\big\|_{B_\infty^{\sigma,\infty}} < \infty.$$

(B) *For $\sigma\in\mathbb{R}$ and $k\in\mathbb{N}$, f belongs to $B_\infty^{\sigma,k,\infty}$ if and only if there exist F and G in $B_\infty^{\sigma,\infty}$ such that $f=F+x_n^kG$. Moreover, the norm of f in $B_\infty^{\sigma,k,\infty}$ is equivalent to*

$$\min_{f=F+x_n^kG}\|F\|_{B_\infty^{\sigma,\infty}}+\|G\|_{B_\infty^{\sigma,\infty}}.$$

Proof. We start from a classical property of Besov spaces: for $g\in C^m$ with $m>|\sigma|$ and $f\in B_\infty^{\sigma,\infty}$, we have

$$\|fg\|_{B_\infty^{\sigma,\infty}}\leqslant C_{m,\sigma}\|f\|_{B_\infty^{\sigma,\infty}}\sup_{|\alpha|\leqslant m}\|\partial^\alpha g\|_\infty.$$

In particular, the norm of f in $B_\infty^{\sigma,\infty}$ is equivalent to $\sup_{j\geqslant 0}\|\eta(x_n/2^j)f(x)\|_{B_\infty^{\sigma,\infty}}$, or equivalently to $\|\eta(x_n)f(x)\|_{B_\infty^{\sigma,\infty}}+\sup_{j\geqslant 0}\|(\eta(x_n/2^{j+1})-\eta(x_n/2^j))f(x)\|_{B_\infty^{\sigma,\infty}}$. Since we have obviously for a positive constant A which does not depend on j

$$\sup_{|\alpha|\leqslant m}\left\|\partial^\alpha\left(\left(\eta\left(x_n/2^{j+1}\right)-\eta\left(x_n/2^j\right)\right)\left(1+x_n^2\right)^{k/2}\right)\right\|_\infty\leqslant A2^{jk}$$

and

$$\sup_{|\alpha|\leqslant m}\left\|\partial^\alpha\left(\left(\eta\left(x_n/2^{j+1}\right)-\eta(x_n/2^j)\right)\left(1+x_n^2\right)^{-k/2}\right)\right\|_\infty\leqslant A2^{-jk},$$

point (A) is easily checked.

Point (B) is easily checked as well: it is enough to write

$$f(x)=\frac{1}{1+x_n^{2k}}f(x)+x_n^k\frac{x_n^k}{1+x_n^{2k}}f(x).\qquad\square$$

The role of tangential Besov spaces is to grant some size control on the distributions we shall deal with. More precisely, we have the following analogous to Lemma 3.

Lemma 5. *Let τ_y be the translation operator $\tau_yf(x)=f(x-y)$ and let δ_λ be the dilation operator $\delta_\lambda f(x)=f(x/\lambda)$. Then*

(A) *Let $\sigma\in\mathbb{R}$ and $k\in\mathbb{R}$. Then*
  (i) *we have for every $y'\in\mathbb{R}^{n-1}$ and every $f\in B_\infty^{\sigma,k,\infty}$,*

$$\|\tau_{(y',0)}f\|_{B_\infty^{\sigma,k,\infty}}=\|f\|_{B_\infty^{\sigma,k,\infty}};$$

  (ii) *for some constant $A_1\geqslant 1$ (depending on k and σ) we have for every $y_n\in[0,1]$ and every $f\in B_\infty^{\sigma,k,\infty}$,*

$$A_1^{-1}\|f\|_{B_\infty^{\sigma,k,\infty}}\leqslant\|\tau_{(0,y_n)}f\|_{B_\infty^{\sigma,k,\infty}}\leqslant A_1\|f\|_{B_\infty^{\sigma,k,\infty}};$$

  (iii) *for some constant $A_2\geqslant 1$ (depending on k and σ) we have for every $\lambda\in[1,2]$ and every $f\in B_\infty^{\sigma,k,\infty}$,*

$$A_2^{-1}\|f\|_{B_\infty^{\sigma,k,\infty}}\leqslant\|\delta_\lambda f\|_{B_\infty^{\sigma,k,\infty}}\leqslant A_2\|f\|_{B_\infty^{\sigma,k,\infty}}.$$

(B) *Conversely, let E be a Banach space which is continuously included in $\mathcal{D}'(\mathbb{R}^n)$. If E is stable under the translations τ_y and the dilations δ_λ and if we have moreover*

(i) *for some constant* $A_0 \geqslant 1$ *we have for every* $y' \in \mathbb{R}^{n-1}$ *and every* $f \in E$,

$$A_0^{-1}\|f\|_E \leqslant \|\tau_{(y',0)}f\|_E \leqslant A_0\|f\|_E;$$

(ii) *for some constant* $A_1 \geqslant 1$ *we have for every* $y_n \in [0,1]$ *and every* $f \in E$,

$$A_1^{-1}\|f\|_E \leqslant \|\tau_{(0,y_n)}f\|_E \leqslant A_1\|f\|_E;$$

(iii) *for some constant* $A_2 \geqslant 1$ (*depending on* k *and* σ) *we have for every* $\lambda \in [1,2]$ *and every* $f \in E$,

$$A_2^{-1}\|f\|_E \leqslant \|\delta_\lambda f\|_E \leqslant A_2\|f\|_E,$$

then E *is embedded in* $B_\infty^{s,k,\infty}$ *for some* $k \geqslant 0$ *and* $\sigma \in \mathbb{R}$.

Proof. Point (A) is easy. We prove point (B). We mimic the proof of Lemma 3: since E is embedded into $\mathcal{D}'(\mathbb{R}^n)$, there exists $N \in \mathbb{N}$ and $C_0 \geqslant 0$ such that for all $\gamma \in \mathcal{D}$ supported in the unit square $(-1,1)^n$ and all $f \in E$ we have

$$\bigl|\langle f \mid \gamma\rangle\bigr| \leqslant C_0\|f\|_E \sum_{|\alpha|\leqslant N} \bigl\|\partial^\alpha\gamma\bigr\|_\infty.$$

We choose $\theta \in \mathcal{D}(\mathbb{R})$ supported in $(-1,1)$ such that $\sum_{k\in\mathbb{Z}}\theta(t-k) = 1$ and we choose $\omega \in \mathcal{D}(\mathbb{R}^n)$ which is supported in $(-1,1)^n$ and satisfies $\sum_{k\in\mathbb{Z}^n}\omega(x-k) = 1$. We write, for $\gamma \in \mathcal{D}(\mathbb{R}^n)$ and $l \in \mathbb{R}$,

$$\begin{aligned}
&\bigl\|\bigl(f(x)\theta(x_n-l)\bigr) * \gamma(x)\bigr\|_\infty \\
&\quad = \sup_{x\in\mathbb{R}^n}\biggl|\sum_{k\in\mathbb{Z}^n}\bigl\langle f(y) \mid \theta(y_n-l)\omega(y-k)\gamma(x-y)\bigr\rangle\biggr| \\
&\quad \leqslant C_0 \sup_{x\in\mathbb{R}^n} \sum_{k\in\mathbb{Z}^n,\,|k_n-l|\leqslant 2} \bigl\|f\bigl(y+(k',l)\bigr)\bigr\|_E \\
&\qquad \times \sum_{|\alpha|\leqslant N}\bigl\|\partial_y^\alpha\bigl(\theta(y_n)\omega\bigl(y'-(0,k_n-l)\bigr)\gamma\bigl(x-y-(k',l)\bigr)\bigr)\bigr\|_\infty \\
&\quad \leqslant C_1\bigl\|f\bigl(y+(0,l)\bigr)\bigr\|_E \sup_{x\in\mathbb{R}^n}\sum_{k'\in\mathbb{Z}^{n-1}}\sum_{|\alpha|\leqslant N}\sup_{y\in(-3,3)^n}\bigl|\bigl(\partial^\alpha\gamma\bigr)\bigl(x-y-(k',l)\bigr)\bigr| \\
&\quad \leqslant C_2\bigl\|f\bigl(y+(0,l)\bigr)\bigr\|_E\sum_{|\alpha|\leqslant N}A\bigl(\partial^\alpha\gamma\bigr).
\end{aligned}$$

Hence, as in Lemma 3, we find that $f\theta(x_n-l)$ belongs to $B_\infty^{-N-n,\infty}$ and that

$$\bigl\|f(x)\theta(x_n-l)\bigr\|_{B_\infty^{-N-n,\infty}} \leqslant C\bigl\|f\bigl(x+(0,l)\bigr)\bigr\|_E.$$

Thus, we can finish the proof by checking that $x_n \mapsto \|\tau_{(0,-l)}f\|_E$ has a polynomial rate of growth when l goes to ∞. This is easy: for $2^j \leqslant |l| < 2^{j+1}$, we write

$$\|\tau_{(0,-l)}f\|_E = \bigl\|\delta_{2^{j+1}}\tau_{(0,-\frac{l}{2^{j+1}})}\delta_{2^{-j-1}}f\bigr\|_E \leqslant A_1A_2^{2j+2}\|f\|_E \leqslant A_1A_2^2\|f\|_E\,|l|^{\frac{2\ln A_2}{\ln 2}}.$$

Thus, Lemma 5 is proved. □

2. The Littlewood–Paley analysis of the operator $\frac{1}{\partial_n+\sqrt{-\Delta'}}$

In this section, we are going to describe the operator $\frac{1}{\partial_n+\sqrt{-\Delta'}}$ which is a convolution operator whose Fourier symbol is $\frac{1}{i\xi_n+|\xi'|}$:

Definition 4. We define $\sqrt{-\Delta'}$, $\frac{1}{\sqrt{-\Delta'}}$ and $\frac{1}{\partial_n+\sqrt{-\Delta'}}$ as the following Fourier multiplier operators: for $\gamma \in \mathcal{S}(\mathbb{R}^n)$,

- $\mathcal{F}(\sqrt{-\Delta'}\gamma)(\xi) = |\xi'|\hat{\gamma}(\xi)$;
- $\mathcal{F}\big(\frac{1}{\sqrt{-\Delta'}}\gamma\big)(\xi) = \frac{1}{|\xi'|}\hat{\gamma}(\xi)$;
- $\mathcal{F}\big(\frac{1}{\partial_n+\sqrt{-\Delta'}}\gamma\big)(\xi) = \frac{1}{i\xi_n+|\xi'|}\hat{\gamma}(\xi)$.

We shall be more precisely interested in the operators of order 1 ∂'_j and $\partial'_j\partial'_k\frac{1}{\sqrt{-\Delta'}}$ and the operators of order 0 $\frac{1}{\partial_n+\sqrt{-\Delta'}}\partial'_j$ and $\frac{1}{\partial_n+\sqrt{-\Delta'}}\partial'_j\partial'_k\frac{1}{\sqrt{-\Delta'}}$ for $1 \leqslant j,k \leqslant n-1$. This includes as well the operator $\sqrt{-\Delta'} = -\sum_{j=1}^{n-1}{\partial'_j}^2\frac{1}{\sqrt{-\Delta'}}$ and the operator $\frac{1}{\partial_n+\sqrt{-\Delta'}}\sqrt{-\Delta'}$.

Lemma 6. *The operators ∂'_j and $\partial'_j\partial'_l\frac{1}{\sqrt{-\Delta'}}$ for $1 \leqslant j,l \leqslant n-1$ maps boundedly the space $B_\infty^{\sigma,\infty}$ to $B_\infty^{\sigma-1,\infty}$ for all $\sigma \in \mathbb{R}$. The same results holds for $*$-weak topologies, from $(B_\infty^{\sigma,\infty}, \sigma(B_\infty^{\sigma,\infty}, B_1^{-\sigma,1}))$ to $(B_\infty^{\sigma-1,\infty}, \sigma(B_\infty^{\sigma-1,\infty}, B_1^{-\sigma+1,1}))$.*

Proof. It is enough to check that those convolution operators are bounded on the preduals: they map $B_1^{-\sigma+1,1}$ to $B_1^{-\sigma,1}$, where the Besov space $B_1^{s,1}$ is defined by

$$f \in B_1^{s,1} \quad \Leftrightarrow \quad S_0 f \in L^1 \quad \text{and} \quad \sum_{j\in\mathbb{N}} 2^{js}\|\Delta_j f\|_1 < \infty.$$

Let $\alpha(t)$ be the inverse Fourier transform of the function $\phi(\tau/4)$ where ϕ is the function in Definition 1. Let $\alpha_j(x_n) = 2^j\alpha(2^j x_n)$ and $\beta_j(x') = 2^{j(n-1)}\alpha(2^j x_1)\ldots\alpha(2^j x_{n-1})$. If $f \in \mathcal{S}$ and if T is one of the operators ∂'_j or $\partial'_j\partial'_l\frac{1}{\sqrt{-\Delta'}}$, we have

$$\Delta_j(Tf) = T(\beta_j \otimes \alpha_j) * \Delta_j f \quad \text{and} \quad S_0(Tf) = T(\beta_0 \otimes \alpha_0) * S_0 f.$$

Thus, we may prove Lemma 6 by checking that

$$\sup_{j\in\mathbb{N}} 2^{-j}\big\|T(\beta_j \otimes \alpha_j)\big\|_1 < \infty.$$

Since T is homogeneous of degree 1, it is equivalent to check that $T(\beta_0 \otimes \alpha_0) \in L^1$. This is easily checked, by writing that

$$T(\beta_0 \otimes \alpha_0) = \big(T_1(x') * \beta_0\big) \otimes \alpha_0 + \big(T_2(x') * \beta_0\big) \otimes \alpha_0,$$

where $T_1 \in \mathcal{E}'(\mathbb{R}^{n-1})$ is a compactly supported distribution (so that $T_1(x') * \beta_0$ belongs to $\mathcal{S}(\mathbb{R}^{n-1})$) and T_2 is a integrable kernel ($T_2 \in L^1(\mathbb{R}^{n-1})$). $\square$

Proposition 1.

(A) *Let K be the inverse Fourier transform of the distribution $\frac{1}{i\xi_n+|\xi'|}$. Then K is supported in the closed half space $\bar{\Omega}$.*

(B) *Let $\eta \in \mathcal{D}(\mathbb{R})$ be equal to 1 on $[-1,1]$. Then, for every $R \geqslant 1$, the convolution with $T(\eta(x_n/R)K(x))$ where T is one of the operators ∂'_j or $\partial'_j\partial'_l\frac{1}{\sqrt{-\Delta'}}$ $(1 \leqslant j,\ l \leqslant n-1)$ maps boundedly the space $B_\infty^{\sigma,\infty}$ to $B_\infty^{\sigma-1/2,\infty}$ for all $\sigma \in \mathbb{R}$. The same results holds for $*$-weak topologies, from $(B_\infty^{\sigma,\infty}, \sigma(B_\infty^{\sigma,\infty}, B_1^{-\sigma,1}))$ to $(B_\infty^{\sigma-1/2,\infty}, \sigma(B_\infty^{\sigma-1/2,\infty}, B_1^{-\sigma+1/2,1}))$.*
Moreover, the convolution operators $f \mapsto \frac{1}{\sqrt{R}}T(\eta(x_n/R)K(x)) * f$ $(R \geqslant 1)$ are equicontinuous from $B_\infty^{\sigma,\infty}$ to $B_\infty^{\sigma-1/2,\infty}$ for all $\sigma \in \mathbb{R}$.

Proof. Since $\frac{1}{i\xi_n+|\xi'|} \in L^1 + L^\infty$, we have for all $\omega \in \mathcal{S}(\mathbb{R}^n)$ that

$$\int \left|\hat{\omega}(\xi)\right| \frac{1}{|i\xi_n + |\xi'||}\,d\xi < \infty$$

so that

$$\langle K \mid \omega\rangle = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^{n-1}} \left(\int\limits_{\mathbb{R}} \hat{\omega}(\xi',\xi_n) \frac{1}{-i\xi_n + |\xi'|} d\xi_n \right) d\xi'$$

(where, formally, $\langle K \mid \omega\rangle = \int \bar{K}(x)\ \omega(x)\,dx = \frac{1}{(2\pi)^n}\int \bar{\hat{K}}(\xi)\hat{\omega}(\xi)\,d\xi$). But we know that the Fourier transform of $1_{t>0}e^{-t}$ is equal to $\frac{1}{1+i\tau}$ so that (defining $\mathcal{F}_{n-1}\omega(\xi',x_n) = \int \omega(x',x_n)e^{-i\xi'.x'}\,dx'$) we find that

$$\frac{1}{2\pi} \int\limits_{\mathbb{R}} \hat{\omega}(\xi',\xi_n) \frac{1}{-i\xi_n + |\xi'|}\,d\xi_n = \int\limits_{x_n>0} \mathcal{F}_{n-1}\omega(\xi',x_n)\ e^{-x_n|\xi'|}\,dx_n.$$

Thus, (A) is proved.

In order to prove (B), it is enough to check that

$$\sup_{R\geqslant 1}\sup_{j\in\mathbb{N}} 2^{-j/2}R^{1/2}\left\|T(\beta_j \otimes \alpha_j) * \left(\eta(x_n/R)K(x)\right)\right\|_1 < \infty.$$

Let $\mu(\xi')$ be the symbol of the operator T ($\mu(\xi') = i\xi_j$ or $\mu(\xi') = -\xi_j\xi_l/|\xi'|$). We define $H(x)$ as the inverse Fourier transform of $\phi(\xi_1/4)\ldots\phi(\xi_{n-1}/4)\mu(\xi')\frac{1}{i\xi_n+|\xi'|}$. We then have (since $\mu(\xi')\frac{1}{i\xi_n+|\xi'|}$ is homogeneous of degree 0)

$$T(\beta_j \otimes \alpha_j) * \left(\eta(x_n/R)K(x)\right) = \int \alpha_j(x_n - y_n)\eta(y_n/R)2^{j(n-1)}H\left(2^j x', 2^j y_n\right)\,dy_n$$

and we find that

$$\begin{aligned}\left\|T(\beta_j \otimes \alpha_j) * \left(\eta\left(\frac{x_n}{R}\right)K(x)\right)\right\|_1 &\leqslant C\left\|\eta\left(\frac{x_n}{R}\right)2^{jn}H\left(2^j x\right)\right\|_1 \\ &\leqslant C'\sqrt{R}\left\|2^{jn}H\left(2^j x\right)\right\|_{L^2_{x_n}L^1_{x'}}.\end{aligned}$$

We have

$$\left\|2^{jn}H\left(2^j x\right)\right\|_{L^2_{x_n}L^1_{x'}} = 2^{j/2}\|H\|_{L^2_{x_n}L^1_{x'}}$$

so that we have just to check that H belongs to $L^2_{x_n}L^1_{x'}$. Since $H \in L^2$, we may write $H = \sum_{j\in\mathbb{Z}} \Delta'_j H = \sum_{j\leqslant 3} \Delta'_j H$. We have $\Delta'_j H = T(\Delta'_j K) *_{\mathbb{R}^{n-1}} \beta_0$, hence $\|\Delta'_j H\|_{L^2_{x_n}L^1_{x'}} \leqslant \|\beta_0\|_1 \|T(\Delta'_j K)\|_{L^2_{x_n}L^1_{x'}}$. One more time, due to homogeneity, we have $\|T(\Delta'_j K)\|_{L^2_{x_n}L^1_{x'}} = 2^{j/2}\|T(\Delta'_0 K)\|_{L^2_{x_n}L^1_{x'}}$. But we have obviously that for all $\alpha \in \mathbb{N}^{n-1}$ the distribution $x'^{\alpha} T(\Delta'_0 K) \in L^2$, as can be checked by the Plancherel equality. In particular,if $N \in \mathbb{N}$ satisfies $(n-1)/2 < N$, we have

$$\int \left(1 + x'^2\right)^N \left|T\left(\Delta'_0 K\right)(x)\right|^2 dx < \infty.$$

Thus, we find that $T(\Delta'_0 K) \in L^2_{x_n}L^1_{x'}$, and we get that H belongs to $L^2_{x_n}L^1_{x'}$. $\square$

3. Weakly singular distributions on the half space

In this section, we study the behaviour of the distributions of the form $\chi_\Omega e^{t\Delta} f$ which will play a prominent role in Ukai's formula. Here, χ_Ω is the characteristic function of Ω and $e^{t\Delta}$ is the well-known heat kernel on $\mathbb{R}^n$:

$$e^{t\Delta} f = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{x^2}{4t}} * f.$$

We shall use as well the restriction operator ρ to Ω:

$$\rho f = f|_\Omega.$$

We thus introduce the following class of distributions on the half space:

Definition 5 *(Weakly singular distributions on the half space).* A distribution T on Ω is weakly singular if there exists $S \in \mathcal{S}'(\mathbb{R}^n)$ such that

(i) $T = \rho S$;
(ii) for some $\sigma \in \mathbb{R}$ and some $k \in \mathbb{N}$, $S \in B^{\sigma,k,\infty}_\infty$;
(iii) for some $\sigma \in \mathbb{R}$ and some $k \in \mathbb{N}$, $\chi_\Omega e^{t\Delta} S$ is bounded in $B^{\sigma,k,\infty}_\infty$ for $0 < t < 1$ and converges in $\mathcal{S}'$ to S as $t \to 0$.

We have the following important lemma which allows us to view weakly singular distributions on Ω as distributions on the whole space that are supported in $\bar{\Omega}$:

Lemma 7. *Let T be a weakly singular distributions on the half space. Then the distribution S satisfying points* (i), (ii) *and* (iii) *in Definition* 5 *is unique.*

Proof. Assume that we have two distributions S_1 and S_2 associated to T. We may choose the same Besov exponent σ and the same growth exponent k for S_1 and S_2 in Definition 5, since $B^{\sigma,\infty}_\infty \subset B^{\tau,\infty}_\infty$ for $\tau \leqslant \sigma$ and since the multiplication by $(1+x_n^2)^{-\alpha}$ is bounded on every $B^{\sigma,\infty}_\infty$ for every non-negative α. Thus, we find that $S = S_1 - S_2$ satisfies

$$\operatorname{Supp} S \subset \partial\Omega, \quad S \in B^{\sigma,\infty}_\infty, \qquad \text{and} \qquad S = \lim \chi_\Omega e^{t\Delta} S \quad \text{in } \mathcal{S}'.$$

We then find (from $\operatorname{Supp} S \subset \partial\Omega$) that

$$S = \sum_{k=0}^{\infty} U_k(x') \otimes \partial_n^k \delta(x_n),$$

where $(U_k)_{k\in\mathbb{N}}$ is a locally finite family of distributions in $\mathbb{R}^{n-1}$. From $S \in B^{\sigma,\infty}_\infty$, we conclude that we have more precisely

$$S = \sum_{0\leqslant k\leqslant -\sigma-1} U_k(x') \otimes \partial_n^k \delta(x_n)$$

with $U_k \in B^{\sigma+k+1,\infty}_\infty$ if $\sigma+k+1<0$ and $U_k \in L^\infty$ if $\sigma+k+1=0$: this is checked by writing that, for $\sigma<0$, we have for all $\varphi \in \mathcal{D}(\mathbb{R})$ and $\Phi \in \mathcal{D}(\mathbb{R}^{n-1})$ and all $\lambda \geqslant 1$ and $\mu \geqslant 1$,

$$\bigl|\bigl\langle S \mid \Phi(\lambda x') \otimes \varphi(\mu x_n)\bigr\rangle\bigr| \leqslant C(\varphi,\Phi)\lambda^{1-n}\mu^{-1}(\lambda+\mu)^{-\sigma}.$$

Next, we write that $\lim_{t\to 0}\langle \chi_\Omega e^{t\Delta} S \mid \Phi(x') \otimes \varphi(x_n)\rangle = \langle S \mid \Phi(x') \otimes \varphi(x_n)\rangle$ and thus, defining

$$H_t(x_n) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x_n^2}{4t}} = \frac{1}{\sqrt{t}} H\left(\frac{x}{\sqrt{t}}\right),$$

$$\lim_{t\to 0} \sum_{0\leqslant k\leqslant -1-\sigma} \bigl\langle \chi_\Omega \partial_n^k H_t(x_n) \bigm| \varphi(x_n)\bigr\rangle \bigl\langle e^{t\Delta'} U_k(x') \bigm| \Phi(x')\bigr\rangle$$
$$= \sum_{0\leqslant k\leqslant -1-\sigma} (-1)^k \partial_n^k \varphi(0) \bigl\langle U_k(x') \bigm| \Phi(x')\bigr\rangle.$$

Since

$$\bigl\langle \chi_\Omega \partial_n^k H_t(x_n) \bigm| \varphi(x_n)\bigr\rangle$$
$$= -t^{-k/2} H^{(k-1)}(0)\varphi(0) + t^{-(k-1)/2} H^{(k-2)}(0)\varphi'(0) + O\bigl(t^{-(k-2)/2}\bigr)$$

and since $H^{(2p+1)}(0) = 0$ and $H^{(2p)}(0) = \frac{1}{\sqrt{4\pi}}(-1)^p \frac{(2p)!}{p!} \neq 0$ for all $p \in \mathbb{N}$, we see that choosing φ_1 with $\varphi_1(0) = 1$ and $\varphi_1'(0) = 0$ and φ_2 with $\varphi_2(0) = 0$ and $\varphi_2'(0) = 1$ and choosing Φ_k such that $\langle U_k(x') \mid \Phi(x')\rangle = 1$ (if $U_k \neq 0$), it is enough to write the asymptotic analysis of the functions $\alpha_{k,i}(t) = \langle \chi_\Omega e^{t\Delta} S \mid \Phi_k(x') \otimes \varphi_i(x_n)\rangle$ to get that $U_k = 0$ for $k \geqslant 1$. Finally, we have

$$\lim_{t\to 0}\bigl\langle \chi_\Omega e^{t\partial_n^2}\delta(x_n) \bigm| \varphi(x_n)\bigr\rangle = \frac{1}{2}\varphi(0) = \frac{1}{2}\bigl\langle \delta(x_n) \mid \varphi(x_n)\bigr\rangle$$

so that $U_0 = 0$ as well. $\square$

Thus, we may introduce the following notations:

Notations.

(i) WS_Ω is the space of weakly singular distributions on the half space.
(ii) For $T \in WS_\Omega$, $\epsilon_0 T$ is the distribution $S \in \mathcal{S}'(\mathbb{R}^n)$ associated to T in Definition 5.
(iii) $WS(\Omega) = \epsilon_0(WS_\Omega)$.
(iv) For $T \in WS_\Omega$, $\epsilon_- T = (\epsilon_0 T)(x', x_n) - (\epsilon_0 T)(x', -x_n)$.

Our definition of weak singularity emphasizes the behaviour of the distributions near the boundary of Ω. Indeed, when a distribution S vanishes near $\partial\Omega$, we may easily control the behaviour of $\chi_\Omega e^{t\Delta} S$:

Proposition 2. *If $T = \rho S$ where $S \in B_\infty^{\sigma,k,\infty}$ for some $k \in \mathbb{N}$ and some $\sigma \in \mathbb{R}$ and* Supp $S \subset$ *$\{x \mid x_n \geqslant \alpha\}$ for some positive α, then T is weakly singular.*

Proof. We see easily that the heat kernel is bounded on $B_\infty^{\sigma,k,\infty}$. Indeed, $U \in B_\infty^{\sigma,k,\infty}$ if and only if U may be written as $U = V + x_n^k W$ with V and $W \in B_\infty^{\sigma,\infty}$ (with $\|V\|_{B_\infty^{\sigma,\infty}} + \|W\|_{B_\infty^{\sigma,\infty}} \leqslant C\|(1+x_n^2)^{-k/2}U\|_{B_\infty^{\sigma,\infty}}$). But we have, defining by $W_{j,t}$ the convolution kernel $\frac{1}{t^{n/2}} W_k(\frac{x}{\sqrt{t}})$ with $W_j = \frac{x_n^j}{(4\pi)^{n/2}} e^{-\frac{x^2}{4}}$,

$$e^{t\Delta}U = e^{t\Delta}V + \sum_{j=0}^{k} \binom{k}{j} t^{j/2} x_n^{k-j} W_{j,t} * W.$$

Thus, for $0 < t < 1$,

$$\left\|\left(1+x_n^2\right)^{-k/2} e^{t\Delta} U\right\|_{B_\infty^{\sigma,\infty}} \leqslant C\left\|\left(1+x_n^2\right)^{-k/2} U\right\|_{B_\infty^{\sigma,\infty}}.$$

Moreover, we may assume that $\sigma < 0$ and write

$$\left\|\left(1+x_n^2\right)^{-k/2}(1-\chi_\Omega)e^{t\Delta}S\right\|_{B_\infty^{\sigma,\infty}} \leqslant C\left\|\left(1+x_n^2\right)^{-k/2}(1-\chi_\Omega)e^{t\Delta}S\right\|_\infty.$$

When $x_n < 0$ and S is supported in $\{y \mid y_n \leqslant \alpha\}$, choosing $\theta \in \mathcal{D}(\mathbb{R})$ equal to 1 on $[-1,1]$ and supported in $[-2,2]$, we may write $e^{t\Delta}S(x) = K_{t,\alpha} * S(x)$ with

$$K_{t,\alpha}(y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{y^2}{4t}}\left(1-\theta\left(\frac{4y}{\alpha}\right)\right).$$

Writing $S = V + x_n^k W$, we find that, for $N > -\sigma$, we have

$$\begin{aligned} &\left\|\left(1+x_n^2\right)^{-k/2}(1-\chi_\Omega)e^{t\Delta}S\right\|_\infty \\ &\quad \leqslant C\left\|\left(1+x_n^2\right)^{-k/2}S\right\|_{B_\infty^{\sigma,\infty}} \sum_{j=0}^{k} \sum_{|\alpha|\leqslant N} \left\|\partial^\alpha\left(x_n^j K_{t,\alpha}(x)\right)\right\|_1 \end{aligned}$$

so that, for some positive constant C which depends on k, N and α, we have

$$\left\|\left(1+x_n^2\right)^{-k/2}(1-\chi_\Omega)e^{t\Delta}S\right\|_{B_\infty^{\sigma,\infty}} = O\left(e^{-\frac{C}{\sqrt{t}}}\right) \quad \text{as } t \to 0.$$

Thus, we find that $\chi_\Omega e^{t\Delta}S$ converges to S in $\mathcal{S}'$. $\square$

When we want to consider distributions whose support meets the border of Ω, we must analyse more carefully the role of χ_Ω. This will be done in the frame of shift-invariant Banach spaces of distributions.

Definition 6 *(Shift-invariant Banach spaces of distributions).*

(A) A shift-invariant Banach space of test functions is a Banach space E such that we have the continuous embeddings $\mathcal{D}(\mathbb{R}^n) \subset E \subset \mathcal{D}'(\mathbb{R}^n)$ and such that moreover:
   (a) for all $x_0 \in \mathbb{R}^n$ and for all $f \in E$, $f(x - x_0) \in E$ and $\|f\|_E = \|f(x - x_0)\|_E$.
   (b) $\mathcal{D}(\mathbb{R}^n)$ is dense in E.

(B) A shift-invariant Banach space of distributions is a Banach space E which is the topological dual of a shift-invariant Banach space of test functions $E^{(*)}$.

Remark. An easy consequence of hypotheses (a) and (b) is that a shift-invariant Banach space of test functions E satisfies $\mathcal{S}(\mathbb{R}^n) \subset E \subset \mathcal{S}'(\mathbb{R}^n)$. Similarly, we have for a shift-invariant Banach space of distributions E that $\mathcal{S}(\mathbb{R}^n) \subset E \subset \mathcal{S}'(\mathbb{R}^n)$. A direct consequence of Lemma 3 is that a shift-invariant Banach space of distributions can be more precisely embedded in some Besov space $B^{\sigma,\infty}_\infty$.

Shift-invariant Banach spaces of distributions are adapted to convolution with integrable kernels:

Lemma 8 *(Convolution in shift-invariant spaces of distributions). If E is a shift-invariant Banach space of functions or of distributions and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then for all $f \in E$ we have $f * \varphi \in E$ and $\|f * \varphi\|_E \leqslant \|f\|_E \|\varphi\|_1$.*

Moreover, convolution may be extended into a bounded bilinear operator from $E \times L^1(\mathbb{R}^n)$ to E and we have for all $f \in E$ and all $g \in L^1$ the inequality $\|f * g\|_E \leqslant \|f\|_E \|g\|_1$.

Similarly, partial convolution defines a bounded bilinear operator from $E \times L^1(\mathbb{R}^{n-1})$ to E and we have for all $f \in E$ and all $g \in L^1(\mathbb{R}^{n-1})$ the inequality

$$\left\| \int_{\mathbb{R}^{n-1}} f(x' - y', x_n) g(y') \, dy' \right\|_E \leqslant \|f\|_E \|g\|_1.$$

Proof. It is enough to check that, for $f \in \mathcal{S}(\mathbb{R}^n)$ and $g \in \mathcal{S}(\mathbb{R}^n)$, the Riemann sums $\frac{1}{N^n} \sum_{k \in \mathbb{Z}^n} g(\frac{k}{N}) f(x - \frac{k}{N})$ converge to $f * g$ in $\mathcal{S}(\mathbb{R}^n)$ as N goes to ∞. Similarly, for $f \in \mathcal{S}(\mathbb{R}^n)$ and $g \in \mathcal{S}(\mathbb{R}^{n-1})$, the Riemann sums $\frac{1}{N^{n-1}} \sum_{k \in \mathbb{Z}^{n-1}} g(\frac{k}{N}) f(x' - \frac{k}{N}, x_n)$ converge to $f * g$ in $\mathcal{S}(\mathbb{R}^n)$ as N goes to ∞. $\square$

Definition 7 *(Splitable Banach spaces of distributions).*

(A) A shift-invariant Banach space of test functions E is splitable if:
   (a) for all $f \in E$, $\tilde{f} : x \mapsto f(x', -x_n)$ belongs to E and $\|\tilde{f}\|_E \leqslant \|f\|_E$;
   (b) there exists $C_0 > 0$ such that, for all $f \in \mathcal{D}(\mathbb{R}^n)$, $\chi_\Omega f \in E$ and $\|\chi_\Omega f\|_E \leqslant C_0 \|f\|_E$.

(B) A splitable shift-invariant Banach space of distributions is a Banach space E which is the topological dual of a splitable shift-invariant Banach space of test functions $E^{(*)}$.

Examples of splitable Banach spaces of distributions are L^p ($1 < p \leqslant \infty$) and Besov spaces ($B_p^{s,\infty}$, $1 < p \leqslant \infty$, $-1 + 1/p < s < 0$).

If E is a splitable shift-invariant Banach space of distributions and if $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we check easily that $\varphi = \chi_\Omega \varphi + \tilde{\chi}_\Omega \varphi$ in $E^{(*)}$, and that

$$\varphi = \lim_{\epsilon \to 0} \chi_\Omega(x', x_n - \epsilon)\varphi(x', x_n - \epsilon) + \chi_\Omega(x', -x_n - \epsilon)\varphi(x', x_n + \epsilon) \quad \text{in } E^{(*)},$$

where the function $\chi_\Omega(x', x_n - \epsilon)\varphi(x', x_n - \epsilon)$ is supported in $\{(x', x_n) \mid x_n \geqslant \epsilon\} \subset \Omega$ and the function $\chi_\Omega(x', -x_n - \epsilon)\varphi(x', x_n + \epsilon)$ is supported in $\tilde{\Omega} = \{(x', x_n) \mid x_n < 0\}$. We then define

$$E(\Omega) = \{f \in E \mid \operatorname{Supp} f \subset \bar{\Omega}\} \quad \text{and} \quad E(\tilde{\Omega}) = \{f \in E \mid \operatorname{Supp} f \subset \bar{\tilde{\Omega}}\}$$

and

$$E_\Omega = \rho(E) \quad \text{normed by} \quad \|f\|_{E_\Omega} = \min_{F \in E,\, \rho(F) = f} \|F\|_E.$$

We then have $E = E(\Omega) \oplus E(\tilde{\Omega})$, with $E(\Omega) = \chi_\Omega E$ and $E(\tilde{\Omega}) = \tilde{\chi}_\Omega E$; moreover, ρ is an isomorphism between $E(\Omega)$ and E_Ω. Thus, ϵ_0 is well defined on E_Ω as $\epsilon_0 = \rho^{-1} : E_\Omega \mapsto E(\Omega)$, and ϵ_- is well defined by $\epsilon_- f = \epsilon_0 f + (\epsilon_0 f)\tilde{}$.

Proposition 3. *If $T \in E_\Omega$ where E is a splitable Banach space of distributions, then T is weakly singular.*

Proof. Since E is a shift invariant space of distributions, we have $E \subset B_\infty^{\sigma,\infty}$ for some $\sigma \in \mathbb{R}$. Now, we have $T = \rho S$ with $S \in E(\Omega) \subset B_\infty^{\sigma,\infty}$. Moreover, $\chi_\Omega e^{t\Delta} S$ is bounded in E and converges $*$-weakly to $\chi_\Omega S = S$. $\square$

4. The trace of a weakly singular distribution

We now introduce the trace of a weakly singular distribution.

Definition 8 *(Trace of a weakly singular distribution).* Let $T \in WS_\Omega$ be a weakly singular distribution. If, for some $\sigma \in \mathbb{R}$, $(e^{t\Delta}\epsilon_0 T)(x', 0)$ converges $*$-weakly in $B_\infty^{\sigma,\infty}(\mathbb{R}^{n-1})$ to a distribution $U(x')$, then the trace of T on $\partial\Omega$ is the distribution $2U$ and will be noted as $T|_{\partial\Omega}$ or as $T(x', 0)$.

We may state this definition in an equivalent way by looking at $\chi_\Omega \partial_n e^{t\Delta}\epsilon_0 T$:

Lemma 9. *Let $T \in WS_\Omega$ be a weakly singular distribution. Then the trace of T exists* (*in the sense of Definition* 8) *if and only if, for some $\sigma \in \mathbb{R}$ and some $k \in \mathbb{N}$, $(1 + x_n^2)^{-k/2}\chi_\Omega \partial_n e^{t\Delta}\epsilon_0 T$ converges $*$-weakly in $B_\infty^{\sigma,\infty}(\mathbb{R}^n)$.*

Proof. Since we know that $(1+x_n^2)^{-k'/2}\chi_\Omega e^{t\Delta}\epsilon_0 T$ converges $*$-weakly in $B_\infty^{\sigma',\infty}(\mathbb{R}^n)$ to $(1+x_n^2)^{-k'/2}\epsilon_0 T$ (for some σ' and k'), $(1+x_n^2)^{-(k'+1)/2}\partial_n\chi_\Omega e^{t\Delta}\epsilon_0 T$ converges $*$-weakly in $B_\infty^{\sigma'-1,\infty}(\mathbb{R}^n)$ to $(1+x_n^2)^{-(k'+1)/2}\partial_n\epsilon_0 T$. Moreover, with $k''=\max(k,k')$,

$$\begin{aligned}&\left(1+x_n^2\right)^{-k''/2}\partial_n\chi_\Omega e^{t\Delta}\epsilon_0 T\\&\quad=\left(1+x_n^2\right)^{-k''/2}\chi_\Omega\partial_n e^{t\Delta}\epsilon_0 T+\left(e^{t\Delta}\epsilon_0 T\right)(x',0)\otimes\delta(x_n)\end{aligned}$$

and thus the $*$-weak convergence of $(1+x_n^2)^{-k''/2}\chi_\Omega\partial_n e^{t\Delta}\epsilon_0 T$ in $B_\infty^{\sigma'',\infty}(\mathbb{R}^n)$ with $\sigma''<\min(\sigma,\sigma'-1,-1)$ is equivalent to the $*$-weak convergence of $(e^{t\Delta}\epsilon_0 T)(x',0)\otimes\delta(x_n)$ in $B_\infty^{\sigma'',\infty}(\mathbb{R}^n)$, which in turn is equivalent to the $*$-weak convergence of $(e^{t\Delta}\epsilon_0 T)(x',0)$ in $B_\infty^{\sigma''+1,\infty}(\mathbb{R}^{n-1})$. □

Examples.

(i) If $T=\rho S$ where S is a $\mathcal{C}^1$ function which is bounded with bounded derivatives, then $\epsilon_0 T=\chi_\Omega S$, $\partial_n\epsilon_0 T=\chi_\Omega\partial_n S+S(x',0)\otimes\delta(x_n)$ and thus $\chi_\Omega e^{t\Delta}\partial_n\epsilon_0 T$ converges to $\chi_\Omega\partial_n S+\frac{1}{2}S(x',0)\otimes\delta(x_n)$. Thus, the trace of T is the function $S(x',0)$.

(ii) In the case where $T=\rho S$ where S is supported in $\{x\mid x_n\geqslant\alpha\}$ for some positive α, then $\partial_n T$ is weakly singular and the trace of T is equal to 0.

Not every weakly singular distribution has a trace. As a matter of fact, we may easily characterize those which have a trace:

Proposition 4. *Let $T\in WS_\Omega$. Then the following assertions are equivalent*:

(i) *T has a trace on $\partial\Omega$*;

(ii) *$\partial_n T\in WS_\Omega$.*

Moreover, in that case, we have

$$\partial_n\epsilon_0 T=\epsilon_0\partial_n T+T(x',0)\otimes\delta(x_n). \tag{5}$$

Proof. (i) ⇒ (ii): Let us define $S=\partial_n\epsilon_0 T-T(x',0)\otimes\delta(x_n)$. Then $\rho S=\partial_n T$. We must check the $*$-weak convergence of $(1+x_n)^{-k/2}\chi_\Omega e^{t\Delta}S$ to S in some Besov space $B_\infty^{\sigma,\infty}(\mathbb{R}^n)$. Since $T\in WS_\Omega$ and since T has a trace, we know (from Lemma 9) that we can find k and σ such that we have the *-weak convergence of $(1+x_n)^{-k/2}\partial_n\chi_\Omega e^{t\Delta}\epsilon_0 T$ to $\partial_n\epsilon_0 T$ and of $(1+x_n)^{-k/2}(e^{t\Delta}\epsilon_0 T)(x',0)\otimes\delta(x_n)$ to $\frac{1}{2}T(x',0)\otimes\delta(x_n)$ in the Besov space $B_\infty^{\sigma,\infty}(\mathbb{R}^n)$. On the other hand, we have the convergence of $(1+x_n)^{-k/2}\chi_\Omega e^{t\Delta}(T(x',0)\otimes\delta(x_n))$ to $\frac{1}{2}T(x',0)\otimes\delta(x_n)$. We may then conclude, since

$$\chi_\Omega e^{t\Delta}S=\partial_n\chi_\Omega e^{t\Delta}\epsilon_0 T-\left(e^{t\Delta}\epsilon_0 T\right)(x',0)\otimes\delta(x_n)-\chi_\Omega e^{t\Delta}\left(T(x',0)\otimes\delta(x_n)\right).$$

(ii) ⇒ (i): We have $\partial_n\epsilon_0 T=\epsilon_0\partial_n T+R$ where the distribution R is supported in $\partial\Omega$ In order to prove that $T(x',0)$ is well defined, we need to prove the $*$-weak convergence of $(1+x_n^2)^{-k/2}\chi_\Omega\partial_n e^{t\Delta}\epsilon_0 T$ in $B_\infty^{\sigma,\infty}(\mathbb{R}^n)$ for some k and σ (following Lemma 9). Since

$\partial_n T$ is weakly singular, this is equivalent to the $*$-weak convergence of $\chi_\Omega e^{t\Delta} R$ in some $B_\infty^{\sigma',\infty}(\mathbb{R}^n)$. But we have seen in the previous section (in the proof of Lemma 7) that, if

$$R = \sum_{0\leqslant j\leqslant -\sigma'-1} U_j(x') \otimes \partial_n^j \delta(x_n),$$

then $\chi_\Omega e^{t\Delta} R$ could converge only if the distributions U_j were equal to 0 for $j \geqslant 1$ (in which case we have convergence to $\frac{1}{2} U_0(x') \otimes \delta(x_n)$). Thus, we must prove that $x_n R = 0$.

We have that $x_n R$ is the limit in the sense of distributions of

$$\begin{aligned} x_n \partial_n \chi_\Omega e^{t\Delta} \epsilon_0 T - x_n \chi_\Omega e^{t\Delta} \epsilon_0 \partial_n T &= x_n \chi_\Omega \partial_n e^{t\Delta} \epsilon_0 T - x_n \chi_\Omega e^{t\Delta} \epsilon_0 \partial_n T \\ &= x_n \chi_\Omega e^{t\Delta} R. \end{aligned}$$

But the proof of Lemma 7 gives one more time that $x_n \chi_\Omega e^{t\Delta} R$ can be convergent only if $U_j = 0$ for $j \geqslant 2$, and convergent to R only if $U_1 = 0$ as well (in which case $x_n R = 0$). □

Remark. If $T = \rho S$ with $S \in B_p^{s,\infty}$ and $s > 1/p$ ($1 \leqslant p \leqslant \infty$), we find that T and $\partial_n T$ are weakly singular, hence T has a trace on $\partial\Omega$. This is the usual trace.

In particular, we check easily that the normal component of a divergence-free weakly singular vector field has a trace:

Proposition 5 *(Divergence-free weakly singular vector fields). Let $\vec{u} = (\vec{u}', u_n) \in (WS_\Omega)^n$ be a divergence-free weakly singular vector field*:

$$\nabla.\vec{u} = 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{6}$$

Then:

(i) *the normal component u_n has a trace on $\partial\Omega$;*
(ii) $\nabla.\epsilon_0\vec{u} = 0$ *in* $\mathcal{D}'(\mathbb{R}^n) \Leftrightarrow u_n(x', 0) = 0$ *on* $\partial\Omega$.

Proof. We first notice that the tangential derivatives of a weakly singular distribution are still weakly singular distributions. Thus, since $\partial_n u_n = -\nabla'.\vec{u}'$, $\partial_n u_n$ is weakly singular. Thus, u_n has a trace on $\partial\Omega$. Moreover, we have obviously $\nabla.\epsilon_0\vec{u} = \epsilon_0 \nabla.\vec{u} + u_n(x', 0) \otimes \delta(x_n)$. □

5. Ukai's formalism

We consider the Stokes problem on Ω.

$$\begin{cases} \partial_t \vec{u} = \Delta\vec{u} - \vec{\nabla} p & \text{on } (0, +\infty) \times \Omega, \\ \nabla.\vec{u} = 0 & \text{on } (0, +\infty) \times \Omega, \\ \vec{u}(t, x', 0) = 0 & \text{on } (0, +\infty) \times \partial\Omega, \\ \vec{u}(0, x', x_n) = \vec{u}_0 & \text{on } \Omega, \end{cases} \tag{7}$$

where the initial value $\vec{u}_0 = (\vec{u}_0', u_{0,n})$ satisfies the compatibility conditions

$$\nabla.\vec{u}_0 = 0 \quad \text{on } \Omega, \tag{8}$$

$$u_{0,n}(x',0)=0 \quad \text{on } \partial\Omega. \tag{9}$$

In order to solve (7), Ukai [5] requires moreover the following condition on the pressure p:

$$(\partial_n+\sqrt{-\Delta'})p=0 \quad \text{on } (0,+\infty)\times\Omega. \tag{10}$$

This may be viewed as a requirement of boundedness of the pressure when $x_n\to+\infty$, since we have $\Delta p=0$ on $(0,+\infty)\times\Omega$ as a consequence of (7), thus

$$(\partial_n-\sqrt{-\Delta'})(\partial_n+\sqrt{-\Delta'})p=0,$$

where $\sqrt{-\Delta'}$ is a non-negative operator.

From (7) and (10), defining

$$z=(\partial_n+\sqrt{-\Delta'})u_n=\sqrt{-\Delta'}u_n-\nabla'.\vec{u}',$$

we find

$$\begin{cases} \partial_t z=\Delta z & \text{on } (0,+\infty)\times\Omega, \\ z(t,x',0)=0 & \text{on } (0,+\infty)\times\partial\Omega, \\ z(0,x',x_n)=\sqrt{-\Delta'}u_{0,n}-\nabla'.\vec{u}_0' & \text{on } \Omega, \end{cases} \tag{11}$$

which is solved by

$$z=\rho e^{t\Delta}\epsilon_-\bigl(\sqrt{-\Delta'}u_{0,n}-\nabla'.\vec{u}_0'\bigr), \tag{12}$$

where ϵ_- is the antisymmetric extension operator

$$\epsilon_- f(x',x_n)=\begin{cases} f(x',x_n) & \text{for } x_n>0, \\ -f(x',-x_n) & \text{for } x_n<0. \end{cases}$$

Once z is known, Ukai computes u_n as the solution of

$$\begin{cases} (\partial_n+\sqrt{-\Delta'})u_n=z & \text{on } (0,+\infty)\times\Omega, \\ u_n(t,x',0)=0 & \text{on } (0,+\infty)\times\partial\Omega. \end{cases} \tag{13}$$

Ukai solves (13) by defining

$$h=\epsilon_0 u_n,$$

where ϵ_0 is the extension-by-zero operator

$$\epsilon_0 f(x',x_n)=\begin{cases} f(x,x_n) & \text{for } x_n>0, \\ 0 & \text{for } x_n<0. \end{cases}$$

Since $u_n(t,x',0)=0$ on $(0,+\infty)\times\partial\Omega$, we find that

$$(\partial_n+\sqrt{-\Delta'})h=\epsilon_0(\partial_n+\sqrt{-\Delta'})u_n=\epsilon_0 z,$$

hence

$$u_n=\rho\frac{1}{\partial_n+\sqrt{-\Delta'}}\epsilon_0 z. \tag{14}$$

Thus, we find the following formula for u_n:

$$u_n = \rho\left(\frac{1}{\partial_n + \sqrt{-\Delta'}}\epsilon_0\rho\left(\sqrt{-\Delta'}e^{t\Delta}\epsilon_- u_{0,n}\right)\right) - \rho\left(\frac{1}{\partial_n + \sqrt{-\Delta'}}\epsilon_0\rho\left(\nabla'.e^{t\Delta}\epsilon_-\vec{u}_0'\right)\right). \tag{15}$$

Ukai's last tool is then to introduce

$$\vec{w}' = \vec{u}' + \frac{1}{\sqrt{-\Delta'}}\vec{\nabla}' u_n.$$

Using

$$\vec{\nabla}' p + \frac{1}{\sqrt{-\Delta'}}\vec{\nabla}'\partial_n p = \frac{1}{\sqrt{-\Delta'}}\vec{\nabla}'(\partial_n + \sqrt{-\Delta'})p = 0,$$

we find that

$$\begin{cases} \partial_t \vec{w}' = \Delta\vec{w}' & \text{on } (0,+\infty)\times\Omega, \\ \vec{w}'(t,x',0) = 0 & \text{on } (0,+\infty)\times\partial\Omega, \\ \vec{w}'(0,x',x_n) = \vec{u}_0' + \frac{1}{\sqrt{-\Delta'}}\vec{\nabla}' u_{0,n} & \text{on } \Omega. \end{cases} \tag{16}$$

which is solved by

$$\vec{w}' = \rho e^{t\Delta}\epsilon_-\left(\vec{u}_0' + \frac{1}{\sqrt{-\Delta'}}\vec{\nabla}' u_{0,n}\right). \tag{17}$$

Combining (15) and (17), we find the following formula for $\vec{u}'$:

$$\vec{u}' = \begin{cases} \frac{1}{\sqrt{-\Delta'}}\vec{\nabla}'\rho\left(\frac{1}{\partial_n+\sqrt{-\Delta'}}\epsilon_0\rho(\sqrt{-\Delta'}e^{t\Delta}\epsilon_- u_{0,n})\right) \\ \quad -\frac{1}{\sqrt{-\Delta'}}\vec{\nabla}'\rho\left(\frac{1}{\partial_n+\sqrt{-\Delta'}}\epsilon_0\rho(\vec{\nabla}' e^{t\Delta}\epsilon_-\vec{u}_0')\right) \\ \quad +\rho e^{t\Delta}\epsilon_-\left(\vec{u}_0' + \frac{1}{\sqrt{-\Delta'}}\vec{\nabla}' u_{0,n}\right). \end{cases} \tag{18}$$

The main problem we want to face in this paper is to determine how to use formulas (15) and (18) with singular initial values, i.e., to give sense to formulas (15) and (18) (and to the compatibility conditions (8) and (9) when applied to such singular distributions and to check to which extent we obtain solutions to the Stokes problem (7).

6. Weakly singular initial values for the Stokes problem

Since the tangential operators ∂_j', $\sqrt{-\Delta'}$ or $\frac{1}{\sqrt{-\Delta'}}\partial_j'$ commute with ϵ_0 and ρ and since we may write $\epsilon_0\rho e^{t\Delta} f = \chi_\Omega e^{t\Delta} f$, (15) and (18) may be written as

$$u_n = \rho\left(\frac{1}{\partial_n + \sqrt{-\Delta'}}\sqrt{-\Delta'}\left(\chi_\Omega e^{t\Delta}\epsilon_- u_{0,n}\right) - \frac{1}{\partial_n + \sqrt{-\Delta'}}\nabla'.\left(\chi_\Omega e^{t\Delta}\epsilon_-\vec{u}_0'\right)\right) \tag{19}$$

and

$$\vec{u}' = \rho\left(-\frac{1}{\sqrt{-\Delta'}}\vec{\nabla}'\frac{1}{\partial_n + \sqrt{-\Delta'}}\sqrt{-\Delta'}\left(\chi_\Omega e^{t\Delta}\epsilon_- u_{0,n}\right)\right.$$

$$+ \frac{1}{\sqrt{-\Delta'}} \vec{\nabla}' \frac{1}{\partial_n + \sqrt{-\Delta'}} \nabla'.\big(\chi_\Omega e^{t\Delta} \epsilon_- \vec{u}'_0\big)$$
$$+ \chi_\Omega e^{t\Delta} \epsilon_- \left(\vec{u}'_0 + \frac{1}{\sqrt{-\Delta'}} \vec{\nabla}' u_{0,n} \right) \Bigg). \tag{20}$$

Ukai reformulated (19) and (20) [5] in terms in terms of Riesz transforms. We introduce the following notations:

(i) $\vec{R}$, $\vec{R}'$ and $\vec{S}'$ are the Riesz gradient, the n-dimensional tangential Riesz gradient and the $(n-1)$-dimensional Riesz gradient:

$$\vec{R} = \begin{pmatrix} R_1 \\ \vdots \\ R_n \end{pmatrix}, \qquad \vec{R}' = \begin{pmatrix} R_1 \\ \vdots \\ R_{n-1} \end{pmatrix}, \qquad \vec{S}' = \begin{pmatrix} S'_1 \\ \vdots \\ S'_{n-1} \end{pmatrix}$$

$$\text{with } R_j = \frac{\partial_j}{\sqrt{-\Delta}} \text{ and } S'_j = \frac{\partial'_j}{\sqrt{-\Delta'}}.$$

(ii) $R.$, $R'.$ and $S'.$ are the Riesz divergence, the n-dimensional tangential Riesz divergence and the $(n-1)$-dimensional Riesz divergence

$$R.\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \sum_{j=1}^{n} R_j f_j, \qquad R'.\begin{pmatrix} f_1 \\ \vdots \\ f_{n-1} \end{pmatrix} = \sum_{j=1}^{n-1} R_j f_j,$$
$$S'.\begin{pmatrix} f_1 \\ \vdots \\ f_{n-1} \end{pmatrix} = \sum_{j=1}^{n-1} S'_j f_j.$$

We may then introduce Ukai's formula: using the identity

$$\frac{1}{\partial_n + \sqrt{-\Delta'}} = \frac{\partial_n - \sqrt{-\Delta'}}{\Delta},$$

the solution $\vec{u}$ described in formulas (19) and (20) may be rewritten in terms of those Riesz transforms as

$$u_n = \rho(R_n R'.\vec{S}' - R'.\vec{R}')\big(\chi_\Omega e^{t\Delta} \epsilon_-\big(u_{0,n} - S'.\vec{u}'_0\big)\big) \tag{21}$$

and

$$\vec{u}' = \rho\big(-\vec{S}'(R_n R'.\vec{S}' - R'.\vec{R}')\big(\chi_\Omega e^{t\Delta} \epsilon_-(u_{0,n} - S'.\vec{u}'_0)\big)$$
$$+ \chi_\Omega e^{t\Delta} \epsilon_-\big(\vec{u}'_0 + \vec{S}' u_{0,n}\big)\big). \tag{22}$$

Ukai could then apply (21) and (22) to $\vec{u}_0 \in (L^p(\Omega))^n$ [5] and Cannone, Planchon and Schonbek to $\vec{u}_0 \in (B_p^{n/p-1,\infty}(\Omega))^n$ [2] $(n < p < \infty)$, since the Riesz transforms, the tangential Riesz transforms and the multiplication by χ_Ω operate boundedly on $L^p(\mathbb{R}^n)$ and $B_p^{n/p-1,\infty}(\mathbb{R}^n)$.

As a matter of fact, we may deal with formulas (19) and (20) without any use of the Riesz transforms. We shall prove more precisely the following theorem:

Theorem 1. *Let $\vec{u}_0 \in (WS_\Omega)^n$ satisfy the compatibility conditions*

$$\nabla.\vec{u}_0 = 0 \quad \text{in } \mathcal{D}'(\Omega) \tag{23}$$

and

$$u_{0,n}|_{\partial\Omega} = 0. \tag{24}$$

Define $\vec{U}_0 = (\vec{U}'_0, U_{0,n})$ by $\vec{U}_0 = \epsilon_0 \vec{u}_0$, $e_-\vec{U}_0$ by $e_-\vec{U}_0(x) = \vec{U}_0(x', x_n) - \vec{U}_0(x', -x_n)$ and define $\vec{u}$ as

$$u_n = \rho\left(\frac{1}{\partial_n + \sqrt{-\Delta'}}\sqrt{-\Delta'}\big(\chi_\Omega e^{t\Delta} e_- U_{0,n}\big) - \frac{1}{\partial_n + \sqrt{-\Delta'}}\nabla'.\big(\chi_\Omega e^{t\Delta} e_- \vec{U}'_0\big)\right) \tag{25}$$

and

$$\begin{aligned}\vec{u}' = \rho\bigg(&-\frac{1}{\partial_n + \sqrt{-\Delta'}}\vec{\nabla}'\big(\chi_\Omega e^{t\Delta} e_- U_{0,n}\big)\\ &+ \frac{1}{\partial_n + \sqrt{-\Delta'}}\frac{1}{\sqrt{-\Delta'}}\vec{\nabla}'\big(\nabla'.\big(\chi_\Omega e^{t\Delta} e_- \vec{U}'_0\big)\big)\\ &+ \chi_\Omega e^{t\Delta} e_-\left(\vec{U}'_0 + \frac{1}{\partial_n + \sqrt{-\Delta'}}\vec{\nabla}' U_{0,n} - \frac{1}{\partial_n + \sqrt{-\Delta'}}\frac{1}{\sqrt{-\Delta'}}\vec{\nabla}'\nabla'.\vec{U}'_0\right)\bigg).\end{aligned} \tag{26}$$

Then $\vec{u}$ is a solution of the Stokes problem on Ω associated to the initial value $\vec{u}_0$:

- (i) *$t \mapsto \vec{u}(t,x)$ is a continuous mapping from $(0,+\infty)$ to $(\mathcal{D}'(\Omega))^n$: for all $\vec{\varphi} \in (\mathcal{D}(\Omega))^n$, $t \mapsto \langle \vec{u}(t,x) \mid \vec{\varphi}(x)\rangle_{\mathcal{D}',\mathcal{D}}$ is continuous*;
- (ii) $\lim_{t\to 0} \vec{u}(t,.) = \vec{u}_0$ *in $\mathcal{D}'(\Omega)$*;
- (iii) *there exists $p \in \mathcal{D}'((0,\infty)\times\Omega)$ such that $\partial_t \vec{u} = \Delta\vec{u} - \vec{\nabla} p$ in $(\mathcal{D}'((0,+\infty)\times\Omega))^n$*;
- (iv) $\nabla.\vec{u} = 0$ *in $\mathcal{D}'((0,+\infty)\times\Omega)$*;
- (v) *for all positive t, $x \mapsto \vec{u}(t,x)$ belongs to $(\mathcal{C}(\bar{\Omega}))^n$ and $\vec{u}(t,x',0) = 0$ on $(0,+\infty)\times\partial\Omega$.*

Proof. Let $\vec{\varphi} \in (\mathcal{D}(\Omega))^n$ and let $R \geqslant 1$ such that $\operatorname{Supp}\vec{\varphi} \subset \mathbb{R}^{n-1}\times(0,R)$. Let $\eta \in \mathcal{D}(\mathbb{R})$ such that $\eta(x_n) = 1$ for $|x_n| \leqslant 1$; let K be the inverse Fourier transform of $\frac{1}{i\xi_n + |\xi'|}$; then

$$\begin{aligned}&\big\langle \vec{u}(t,x) \mid \vec{\varphi}(x)\big\rangle_{\mathcal{D}',\mathcal{D}}\\ &\quad= \big\langle \sqrt{-\Delta'}\big(\eta(x_n/R)K(x)\big) * \big(\eta(x_n/R)\chi_\Omega e^{t\Delta} e_- U_{0,n}\big) \bigm| \varphi_n\big\rangle_{\mathcal{D}',\mathcal{D}}\\ &\qquad - \sum_{j=1}^{n-1}\big\langle \partial'_j\big(\eta(x_n/R)K(x)\big) * \big(\eta(x_n/R)\chi_\Omega e^{t\Delta} e_- U'_{0,j}\big) \bigm| \varphi_n\big\rangle_{\mathcal{D}',\mathcal{D}}\\ &\qquad - \sum_{j=1}^{n-1}\big\langle \partial'_j\big(\eta(x_n/R)K(x)\big) * \big(\eta(x_n/R)\chi_\Omega e^{t\Delta} e_- U_{0,n}\big) \bigm| \varphi_j\big\rangle_{\mathcal{D}',\mathcal{D}}\\ &\qquad + \sum_{j=1}^{n-1}\sum_{l=1}^{n-1}\left\langle \frac{1}{\sqrt{-\Delta'}}\partial'_j\partial'_l\big(\eta(x_n/R)K(x)\big) * \big(\eta(x_n/R)\chi_\Omega e^{t\Delta} e_- U'_{0,l}\big) \,\middle|\, \varphi_j\right\rangle_{\mathcal{D}',\mathcal{D}}\end{aligned}$$

$$+\left\langle \eta\left(\frac{x_n}{R}\right)\chi_\Omega e^{t\Delta} e_-\left(\vec{U}_0' + \frac{1}{\partial_n+\sqrt{-\Delta'}}\vec{\nabla}' U_{0,n} - \frac{1}{\partial_n+\sqrt{-\Delta'}}\frac{1}{\sqrt{-\Delta'}}\vec{\nabla}'\nabla'.\vec{U}_0'\right) \,\middle|\, \vec{\varphi}'\right\rangle_{\mathcal{D}',\mathcal{D}}.$$

The heat kernel $e^{t\Delta}$ maps $B_\infty^{\sigma,\infty}$ to $B_\infty^{\tau,\infty}$ for every σ and every $\tau > \sigma$. In particular, since $B_\infty^{\tau,\infty} \subset L^\infty$ for $\tau > 0$, we find that for each $\sigma \in \mathbb{R}$ and $k \in \mathbb{R}$, for each $f \in B_\infty^{\sigma,k,\infty}(\mathbb{R}^n)$ and for each $R \geqslant 1$, the map $t \mapsto \eta(x_n/R)\chi_\Omega e^{t\Delta} f$ is continuous from $(0,\infty)$ to $(L^\infty, \sigma(L^\infty, L^1))$, hence to $(B_\infty^{-\epsilon,\infty}, \sigma(B_\infty^{-\epsilon,\infty}, B_1^{\epsilon,1}))$ for every positive ϵ. Then, due to Proposition 1, we find that for every operator T of the form ∂_j' or $\partial_j'\partial_l'\frac{1}{\sqrt{-\Delta'}}$, the map $t \mapsto T(\eta(x_n/R)K(x)) * \eta(x_n/R)\chi_\Omega e^{t\Delta} f$ is continuous from $(0,\infty)$ to $(B_\infty^{-\epsilon-1/2,\infty}, \sigma(B_\infty^{-\epsilon-1/2,\infty}, B_1^{\epsilon+1/2,1}))$ for every positive ϵ.

Moreover, Proposition 1 shows us that, for every $\sigma \in \mathbb{R}$ and every $k \geqslant 0$, for every $f \in B_\infty^{\sigma,k,\infty}$ which is supported in $\bar{\Omega}$ and for every operator T of the form ∂_j' or $\partial_j'\partial_l'\frac{1}{\sqrt{-\Delta'}}$, the distribution $T(K) * f$ belongs to $B_\infty^{\sigma-1/2,k+1/2,\infty}$.

Thus, (i) is proved.

We now use the assumption of weak singularity. We know that $\chi_\Omega e^{t\Delta}\vec{U}_0$ converges to $\vec{U}_0$. Since $e^{t\Delta}\vec{U}_0$ converges to $\vec{U}_0$, we find that $(1-\chi_\Omega)e^{t\Delta}\vec{U}_0$ converges to 0, hence $\chi_\Omega e^{t\Delta} e_-\vec{U}_0$ converges to $\vec{U}_0$. Using Proposition 1 again, we find that, for $\vec{\varphi} \in (\mathcal{D}(\Omega))^n$, we have

$$\begin{aligned}
&\lim_{t\to 0}\langle \vec{u}(t,x) \mid \vec{\varphi}(x)\rangle_{\mathcal{D}',\mathcal{D}} \\
&\quad= \langle \sqrt{-\Delta'}K * U_{0,n} \mid \varphi_n\rangle_{\mathcal{D}',\mathcal{D}} - \sum_{j=1}^{n-1}\langle \partial_j' K * U_{0,j}' \mid \varphi_n\rangle_{\mathcal{D}',\mathcal{D}} \\
&\qquad- \sum_{j=1}^{n-1}\langle \partial_j' K * U_{0,n} \mid \varphi_j\rangle_{\mathcal{D}',\mathcal{D}} + \sum_{j=1}^{n-1}\sum_{l=1}^{n-1}\left\langle \frac{1}{\sqrt{-\Delta'}}\partial_j'\partial_l' K * U_{0,l}' \,\middle|\, \varphi_j\right\rangle_{\mathcal{D}',\mathcal{D}} \\
&\qquad+ \left\langle \vec{U}_0' + \frac{1}{\partial_n+\sqrt{-\Delta'}}\vec{\nabla}' U_{0,n} - \frac{1}{\partial_n+\sqrt{-\Delta'}}\frac{1}{\sqrt{-\Delta'}}\vec{\nabla}'\nabla'.\vec{U}_0' \,\middle|\, \vec{\varphi}'\right\rangle_{\mathcal{D}',\mathcal{D}}
\end{aligned}$$

(where we simply used for the last term on the right hand of this equality the fact that $\eta(x_n/R)\chi_\Omega(x)\vec{\varphi}'(x) = \vec{\varphi}'(x)$) and thus we have

$$\lim_{t\to 0}\vec{u} = \left(\rho\vec{U}_0', \rho\frac{1}{\partial_n+\sqrt{-\Delta'}}\left(\sqrt{-\Delta'}U_{0,n} - \nabla'.\vec{U}_0'\right)\right).$$

We have (following the proof of Proposition 1)

$$\lim_{j\to-\infty} S_j'\frac{1}{\partial_n+\sqrt{-\Delta'}}\left(\sqrt{-\Delta'}U_{0,n} - \nabla'.\vec{U}_0'\right) = 0$$

and

$$\lim_{j\to-\infty} S_j'\left(\sqrt{-\Delta'}U_{0,n} - \nabla'.\vec{U}_0'\right) = 0$$

while

$$(\partial_n+\sqrt{-\Delta'})(\mathrm{Id}-S'_j)\frac{1}{\partial_n+\sqrt{-\Delta'}}\big(\sqrt{-\Delta'}U_{0,n}-\nabla'.\vec{U}'_0\big)$$
$$=\big(\mathrm{Id}-S'_j\big)\big(\sqrt{-\Delta'}U_{0,n}-\nabla'.\vec{U}'_0\big).$$

If we define

$$V_0=\frac{1}{\partial_n+\sqrt{-\Delta'}}\big(\sqrt{-\Delta'}U_{0,n}-\nabla'.\vec{U}'_0\big),$$

we find that

$$(\partial_n+\sqrt{-\Delta'})V_0=\sqrt{-\Delta'}U_{0,n}-\nabla'.\vec{U}'_0=(\partial_n+\sqrt{-\Delta'})U_{0,n}$$

(since $\nabla.\vec{U}_0=0$ due to the compatibility conditions (23) and (24)). Hence, we find that $\Delta(V_0-U_{0,n})=0$; but $V_0-U_{0,n}$ is a tempered distribution, then it must be a harmonic polynomial. Moreover, we have that $V_0-U_{0,n}$ is supported in the half space $\bar{\Omega}$, thus it must be equal to 0.

Thus, (ii) is proved.

In order to prove (iii), we define $\vec{V}=(\vec{V}',V_n)$ as

$$V_n=\frac{1}{\partial_n+\sqrt{-\Delta'}}\sqrt{-\Delta'}\big(\chi_\Omega e^{t\Delta}e_-U_{0,n}\big)-\frac{1}{\partial_n+\sqrt{-\Delta'}}\nabla'.\big(\chi_\Omega e^{t\Delta}e_-\vec{U}'_0\big)$$

and

$$\vec{V}'=-\frac{1}{\partial_n+\sqrt{-\Delta'}}\vec{\nabla}'\big(\chi_\Omega e^{t\Delta}e_-U_{0,n}\big)+\frac{1}{\partial_n+\sqrt{-\Delta'}}\frac{1}{\sqrt{-\Delta'}}\vec{\nabla}'\big(\nabla'.\big(\chi_\Omega e^{t\Delta}e_-\vec{U}'_0\big)\big).$$

We have clearly

$$(\partial_t-\Delta)\vec{u}=\rho(\partial_t-\Delta)\vec{V}.$$

Moreover, we have

$$\lim_{j\to-\infty}S'_j\vec{V}=0.$$

We then define, for $j\in\mathbb{Z}$, W_j by

$$W_j=\frac{1}{\partial_n+\sqrt{-\Delta'}}\big(\mathrm{Id}-S'_j\big)\big(\chi_\Omega e^{t\Delta}e_-U_{0,n}\big)$$
$$-\frac{1}{\partial_n+\sqrt{-\Delta'}}\frac{1}{\sqrt{-\Delta'}}\nabla'.\big(\mathrm{Id}-S'_j\big)\big(\chi_\Omega e^{t\Delta}e_-U'_0\big),$$

where the operators $\frac{1}{\partial_n+\sqrt{-\Delta'}}(\mathrm{Id}-S'_j)$ and $\frac{1}{\partial_n+\sqrt{-\Delta'}}\frac{1}{\sqrt{-\Delta'}}\nabla'.$ are convolution operators well defined on $\mathcal{S}'$. Since we have

$$\sqrt{-\Delta'}\frac{1}{\partial_n+\sqrt{-\Delta'}}\big(\mathrm{Id}-S'_j\big)=\big(\mathrm{Id}-S'_j\big)-\partial_n\frac{1}{\partial_n+\sqrt{-\Delta'}}\big(\mathrm{Id}-S'_j\big),$$

we find that we have

$$\big(\mathrm{Id}-S'_j\big)V_n=-\partial_nW_j+\big(\mathrm{Id}-S'_j\big)\chi_\Omega e^{t\Delta}e_-U_{0,n}$$
$$-\big(\mathrm{Id}-S'_j\big)\frac{1}{\sqrt{-\Delta'}}\nabla'.\chi_\Omega e^{t\Delta}e_-\vec{U}'_0$$

and

$$\big(\mathrm{Id} - S'_j\big)\vec{V}' = -\vec{\nabla}' W_j$$

and thus we have

$$\rho(\partial_t - \Delta)\big(\mathrm{Id} - S'_j\big)\vec{V} = -\rho\vec{\nabla}(\partial_t - \Delta)W_j.$$

This gives

$$(\partial_t - \Delta)\vec{u} = -\lim_{j\to-\infty} \vec{\nabla}\rho(\partial_t - \Delta)W_j.$$

But a classical result states that $\vec{T} \in (\mathcal{D}'((0,+\infty)\times\Omega))^n$ can be written as $\vec{T} = \vec{\nabla}p$ for some distribution $p \in \mathcal{D}'((0,+\infty)\times\Omega)$ if and only if $\vec{T}$ is orthogonal to the divergence-free test functions:

$$\vec{\omega} \in \big(\mathcal{D}\big((0,+\infty)\times\Omega\big)\big)^n \quad \text{and} \quad \boldsymbol{\nabla}.\vec{\omega} = 0 \Rightarrow \langle\vec{T} \mid \vec{\omega}\rangle = 0.$$

Thus, we find that there exists some distribution p such that

$$\lim_{j\to-\infty} \vec{\nabla}\rho(\partial_t - \Delta)W_j = \vec{\nabla}p$$

and finally

$$(\partial_t - \Delta)\vec{u} = -\vec{\nabla}p.$$

Thus, (iii) is proved.

We now compute the divergence of $\vec{u}$. We have $\vec{u} = \rho(\vec{V} + (\vec{Z}', 0))$ with

$$\vec{Z}' = \chi_\Omega e^{t\Delta} e_-\left(\vec{U}'_0 + \frac{1}{\partial_n + \sqrt{-\Delta'}}\vec{\nabla}' U_{0,n} - \frac{1}{\partial_n + \sqrt{-\Delta'}}\frac{1}{\sqrt{-\Delta'}}\vec{\nabla}'\boldsymbol{\nabla}'.\vec{U}'_0\right).$$

We have $\boldsymbol{\nabla}.\vec{u} = \rho(\boldsymbol{\nabla}.\vec{V} + \nabla'.\vec{Z}')$. We have

$$\lim_{j\to-\infty} S'_j\vec{V} = \lim_{j\to-\infty} S'_j\boldsymbol{\nabla}'.\vec{Z}' = 0.$$

This gives

$$\boldsymbol{\nabla}.\vec{u} = \lim_{j\to-\infty}\rho\big(\boldsymbol{\nabla}.\big(\mathrm{Id} - S'_j\big)\vec{V} + \boldsymbol{\nabla}'.\big(\mathrm{Id} - S'_j\big)\vec{Z}'\big).$$

We have

$$\boldsymbol{\nabla}'.\big(\mathrm{Id} - S'_j\big)\vec{Z}' = \chi_\Omega\left(\big(\mathrm{Id} - S'_j\big)e^{t\Delta}e_-\left(\boldsymbol{\nabla}'.\vec{U}'_0 + \frac{1}{\partial_n + \sqrt{-\Delta'}}\Delta' U_{0,n} - \frac{1}{\partial_n + \sqrt{-\Delta'}}\frac{1}{\sqrt{-\Delta'}}\Delta'\boldsymbol{\nabla}'.\vec{U}'_0\right)\right).$$

We may write

$$-\frac{1}{\partial_n + \sqrt{-\Delta'}}\frac{1}{\sqrt{-\Delta'}}\Delta'\boldsymbol{\nabla}'.\vec{U}'_0 = \frac{1}{\partial_n + \sqrt{-\Delta'}}\frac{1}{\sqrt{-\Delta'}}\Delta'\partial_n U_{0,n}$$

and

$$\frac{1}{\partial_n + \sqrt{-\Delta'}}\Delta' U_{0,n} = \frac{1}{\partial_n + \sqrt{-\Delta'}}\frac{1}{\sqrt{-\Delta'}}\Delta'\sqrt{-\Delta'}U_{0,n}$$

so that

$$\nabla'.\big(\mathrm{Id}-S'_j\big)\vec{Z}' = \chi_\Omega\big(\big(\mathrm{Id}-S'_j\big)e^{t\Delta}e_-\big(\nabla'.\vec{U}'_0 - \sqrt{-\Delta'}U_{0,n}\big)\big).$$

We have $(\mathrm{Id}-S'_j)\vec{V}' = -\frac{1}{\sqrt{-\Delta'}}\vec{\nabla}'(\mathrm{Id}-S'_j)V_n$, so that

$$\begin{aligned}\nabla.\big(\mathrm{Id}-S'_j\big)\vec{V} &= \big(\partial_n+\sqrt{-\Delta'}\big)\big(\mathrm{Id}-S'_j\big)V_n \\ &= \chi_\Omega\big(\mathrm{Id}-S'_j\big)\big(\sqrt{-\Delta'}e^{t\Delta}e_-U_{0,n} - \nabla'.e^{t\Delta}e_-\vec{U}'_0\big).\end{aligned}$$

Thus, we have

$$\big(\mathrm{Id}-S'_j\big)(\nabla.\vec{V}+\nabla'.\vec{Z}') = 0$$

and finally $\nabla.\vec{u}=0$. Thus, (iv) is proved.

We now deal with point (v). The heat kernel $e^{t\Delta}$ maps $B_\infty^{\sigma,k,\infty}$ to $B_\infty^{\tau,k,\infty}$ for every σ and every $\tau>\sigma$. In particular, for every $f\in B_\infty^{\sigma,k,\infty}$ which is supported in $\bar{\Omega}$ and for every $\epsilon\in(0,1/2)$, $e^{t\Delta}e_-f$ belongs to $B_\infty^{1/2+\epsilon,k,\infty}$. If $\eta\in\mathcal{D}(\mathbb{R})$ is equal to 1 in the neighbourhood of 0, we find that

$$\chi_\Omega e^{t\Delta}e_-f = \chi_\Omega\eta(x_n)e^{t\Delta}e_-f + \chi_\Omega\big(1-\eta(x_n)\big)e^{t\Delta}e_-f.$$

The function $\chi_\Omega(1-\eta(x_n))e^{t\Delta}e_-f$ still belongs to $B_\infty^{1/2+\epsilon,k,\infty}$. The function $\eta(x_n)e^{t\Delta}e_-f$ belongs to $B_\infty^{1/2+\epsilon,\infty}$ and is identically equal to 0 on $\partial\Omega$ (due to the fact that the heat kernel is an even function with respect to x_n and e_-f is an odd distribution with respect to x_n). Thus, the function $\chi_\Omega\eta(x_n)e^{t\Delta}e_-f$ still belongs to $B_\infty^{1/2+\epsilon,\infty}$: indeed, let $g=\eta(x_n)e^{t\Delta}e_-f$; since $\epsilon+1/2\in(0,1)$, we just have to check that $\|\chi_\Omega g\|_\infty\leqslant\|g\|_\infty$ and that

$$\big|\chi_\Omega(x)g(x)-\chi_\Omega(y)g(y)\big| \leqslant |x-y|^{1/2+\epsilon}\sup_{a\neq b}\frac{|g(a)-g(b)|}{|a-b|^{1/2+\epsilon}} \quad \text{if } g=0 \text{ on } \partial\Omega;$$

this is easy: we may assume that $x_n>0$ and $y_n<0$; we write $\chi_\Omega(x)g(x)=g(x)$ and $\chi_\Omega(y)g(y)=0=g(y',0)$ with

$$\big|g(x)-g(y',0)\big| \leqslant \big|(x'-y',x_n)\big|^{1/2+\epsilon}\sup_{a\neq b}\frac{|g(a)-g(b)|}{|a-b|^{1/2+\epsilon}}$$

and we conclude since $|(x'-y',x_n)|\leqslant|x-y|$.

Thus, we know that $\chi_\Omega e^{t\Delta}e_-f$ belongs to $B_\infty^{1/2+\epsilon,k,\infty}$ and is supported in $\bar{\Omega}$. Then, due to Proposition 1, we find that for every operator T of the form ∂'_j or $\partial'_j\partial'_l\frac{1}{\sqrt{-\Delta'}}$, the distribution $\frac{1}{\partial_n+\sqrt{-\Delta'}}T(\chi_\Omega e^{t\Delta}e_-f)$ belongs to $B_\infty^{\epsilon,k+1/2,\infty}$ and is supported in $\bar{\Omega}$. In particular, it is continuous and identically equal to 0 on $\partial\Omega$. Thus, (v) is proved. □

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